

HIGH-ORDER COMPARISONS BETWEEN POST-NEWTONIAN AND PERTURBATIVE SELF FORCES

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Recent numerical and analytic computations based on the self-force (SF) formalism in general relativity showed that half-integral post-Newtonian (PN) terms, i.e. terms involving odd powers of $1/c$, arise in the redshift factor of small mass-ratio black-hole binaries on exact circular orbits. Although those contributions might seem puzzling at first sight for conservative systems that are invariant under time-reversal, they are in fact associated with the so-called non-linear tail-of-tail effect. We shall describe here how the next-to-next-to-leading order contributions beyond the first half-integral 5.5PN conservative effect (i.e. up to order 7.5PN included) have been obtained by means of the standard PN formalism applied to binary systems of point-like objects. The resulting redshift factor in the small mass-ratio limit fully agrees with that of the SF approach.

1 Introduction

Stellar-mass compact objects inspiraling gradually about super-massive black holes may produce gravitational waves detectable by future space missions such as eLISA¹. These systems, referred to as Extreme Mass Ratio Inspirals (EMRIs), can probe the strong gravity field regime, but proper data analysis of the resulting signal will require accurate waveform templates built from theoretical models. This has motivated, over the past ten years, numerous studies on the dynamics of point-like objects on a curved background^{2,3,4}. Due to the metric perturbations generated by its own mass-energy, the point mass effectively feels a self-force (SF) that induces deviations from the geodesic worldline followed by a test particle on the background. When the background is a black-hole spacetime, the acceleration may be sought in the form of an asymptotic expansion in powers of the mass ratio $q = m_1/m_2 \ll 1$.

In the SF approach, the first order perturbation $\delta g_{\mu\nu}$ of the background metric $g_{\mu\nu}^{(0)}$ is obtained by convolving over the stress-energy tensor $T^{\mu\nu}$ the regularized (R) Green function $G_{\text{R}}^{\mu\nu}{}_{\alpha'\beta'}(x, x')$ that solves the linearized homogeneous Einstein equations in harmonic gauge and has the property that it coincides with the retarded Green function when x lies in the chronological future of x' , while vanishing when x is in the chronological past of x' ⁵. The trajectory of the particle is then precisely that of a geodesic for the perturbed metric $g_{\mu\nu}^{(0)} + \delta g_{\mu\nu}$.

By contrast, the post-Newtonian (PN) approach is based on the formal expansion, on a flat background spacetime, of all quantities of interest, in powers of v/c , where v is the largest typical velocity of the problem. The standard PN approach is first defined for general extended PN sources with compact support and then specialized to compact binary systems⁶. In that case,

v is taken to be the relative coordinate velocity v_{12} . Moreover, when the bodies are compact, they may be effectively represented as point particles. Ultra-violet (UV) divergences at the particle positions are tackled by means of dimensional regularization^{7,8}. The PN expressions are valid at a given coordinate time in a spatial region, referred to as the near zone, that entirely contains the matter source and whose radius is much smaller than the gravitational wavelength. Because SF and PN methods are so radically different from each other, notably regarding the regularization schemes, comparing PN expansions of observable quantities truncated at linear order in q to their SF counterparts expanded in power of $1/c$ allows for non-trivial cross-checks that strengthen our confidence in both perturbative techniques.

After the first comparison between PN and gravitational SF calculations⁹, rapid progress has been made over the last six years, mainly due to both high precision numerical computations from a SF perspective and extensive analytical PN computations^{10,11}. Recently, after the possibility for this comparison had been dramatically extended from the SF side¹², it was realized that observable quantities could contain half-integral $\frac{n}{2}$ PN terms that are nevertheless conservative, starting at the order 5.5PN¹². Here, we shall explain how those terms can arise within the PN framework¹³ and sketch their calculation at the next-to-next-to-leading order^{13,14}. As we shall see, they are closely related to the so-called tail-of-tail effects in general relativity. The success of this SF/PN comparison actually provides an excellent test of the intricate PN machinery for computing non-linear tail-of-tail effects — these being relevant for template waveform generation of comparable mass compact binaries to be analyzed in ground and space based detectors.

2 Comparing post-Newtonian and self-force results

2.1 The Detweiler variable

We shall focus on the Detweiler variable⁹, which represents physically the inverse of the redshift of a photon emitted by a particle moving on an exact circular orbit around a Schwarzschild black hole and detected by an infinitely far-away observer along the rotation axis. The ensuing spacetime is helically symmetric, with a helical Killing vector K^α tangent to the four-velocity u_1^α on the particle worldline. Alternatively, the redshift variable is defined geometrically as the conserved quantity associated with the helical Killing symmetry relevant to spacetimes with exactly circular orbits. In an appropriate class of coordinate systems, the redshift factor u_1^T reduces to the t component of the particle's four-velocity,

$$u_1^T = \frac{1}{\sqrt{-g_{\alpha\beta}(y_1)v_1^\alpha v_1^\beta/c^2}}, \quad (1)$$

where $g_{\alpha\beta}(y_1)$ denotes the metric evaluated at the particle's location $y_1^\alpha = (ct, y_1^i)$ by means of dimensional regularization, and $v_1^\alpha \equiv dy_1^\alpha/dt = (c, v_1^i)$ is the coordinate velocity.

The Detweiler variable (1) has been computed to high-order using on the one hand standard PN theory supplemented with dimensional regularization, valid in weak field^{10,11}, and on the other hand both numerical and analytical SF approaches, valid in the limit $q = m_1/m_2 \ll 1$. Over the last two years, its accuracy has improved drastically on the SF side due to the new application of methods to represent analytic solutions for metric perturbations of black-hole spacetimes. In a first stage, based on some exact solutions of the Teukolsky equation¹⁵, the PN coefficients of the redshift factor were obtained numerically to 10.5PN order¹². Analytic expressions were even found for a subset of coefficients, specifically those that are either rational, or made of the product of π with a rational, or a sum of commonly occurring transcendentals. An alternative SF approach¹⁶, based on the post-Minkowskian expansion of the Regge-Wheeler-Zerilli equation¹⁷, has also reached PN coefficients analytically up to 8.5PN order. Most recently, both methods have been extended to extremely high orders, typically 21.5PN for the redshift factor^{18,19}.

The appearance of *half-integral* PN coefficients (of type $\frac{n}{2}$ PN with n being an odd integer) in the conservative dynamics of two particles on circular orbits is a feature of high-order PN expansion. Resorting to standard PN methods, we shall show now that the leading half-integral PN terms originate from non-linear integrals depending on the past history of the source — so-called hereditary type integrals.

2.2 Dimensionality argument

The fact that terms at half-integral PN orders cannot stem from the source variables evaluated at the current time follows from the general structure of “instantaneous” terms entering the redshift factor (1) in the center-of-mass frame. After replacing all accelerations by the lower-order equations of motion, u_1^T takes the form (with usual Euclidean notation)

$$(u_1^T)_{\text{inst}} \sim \sum_{\substack{j,k,n,p \\ \text{integers}}} \nu^j \left(\frac{Gm}{r_{12}c^2} \right)^k \left(\frac{\mathbf{v}_{12}^2}{c^2} \right)^n \left(\frac{\mathbf{n}_{12} \cdot \mathbf{v}_{12}}{c} \right)^p, \quad (2)$$

where $m = m_1 + m_2$, $\nu = m_1 m_2 / m^2$, while $\mathbf{n}_{12} = (\mathbf{y}_1 - \mathbf{y}_2) / r_{12}$ stands for the relative direction of the two particles. We have taken the expansion when the mass ratio $\nu \rightarrow 0$.

The counting of the $1/c$ powers shows that the PN order of the generic term in Eq. (2) can be half-integral only if p is odd, in which case it vanishes for circular orbits, when the velocity \mathbf{v}_{12} and unit direction \mathbf{n}_{12} are evaluated *at the same current time* t . However, integration over some intermediate time extending from the infinite past up to t could allow a coupling between these vectors at different times. Of course, in general relativity, this type of “hereditary” dependence over the past of the system does occur, due, in particular, to wave tails produced by the back-scatter of linear waves on the spacetime curvature.

3 Post-Newtonian computation of half-integral order contributions

3.1 Structure of the tail contributions

The tail effects, associated with non-linear wave propagation, are best investigated by constructing the (multipolar-)post-Minkowskian expansion of the metric $g_{\mu\nu}$ in powers of the gravitational constant G , outside the matter source⁶. We start from the Einstein field equations in vacuum, written in terms of the field perturbation variable $h^{\mu\nu} \equiv \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}$ on the flat background $\eta^{\mu\nu}$ in Cartesian coordinates $\{x^i\}$, with g representing the determinant of $g_{\mu\nu}$. Adopting the harmonic-gauge condition $\partial_\nu h^{\mu\nu} = 0$, the relaxed Einstein equations for $h^{\mu\nu}$ reduce to the wave-like equations $\square h^{\mu\nu} = \Lambda^{\mu\nu}$, with $\square \equiv \eta^{\alpha\beta} \partial_\alpha \partial_\beta$. The non-linear source term $\Lambda^{\mu\nu}$ is an expression of second-order (at least) in $h^{\alpha\beta}$. At linear order in G , the most general solution depends on six sets of source multipole moments: the mass-type moments, $I_L \equiv I_{i_1 \dots i_\ell}$ (ℓ being the multipole order), the current-type moments $J_L \equiv J_{i_1 \dots i_\ell}$, and four sets of so-called gauge moments, irrelevant for the present discussion. The higher order solutions are obtained iteratively by applying the flat retarded integral operator $\square_{\text{ret}}^{-1}$ on the source term, after multiplication by a regularization factor r^B to cope with the divergence of the multipole expansion when $r \equiv |\mathbf{x}| \rightarrow 0$. Analytic continuation in $B \in \mathbb{C}$ is invoked and the finite part (FP) when $B \rightarrow 0$ provides a certain particular solution. To ensure that the harmonic coordinate condition is satisfied at each step, we must add to the latter solution a specific homogeneous retarded solution, which does not generate tail integrals and can be safely ignored here.

Since, ultimately, we shall be interested in the metric at the location of one of the particles, our goal is to compute the near-zone expansion, indicated below by an overline, of the general solution initially defined in the exterior of the source, when $r \rightarrow 0$. It is obtained directly at a given order from the near-zone expansion of the corresponding source, without need to control

the full solution, from the formula²⁰:

$$\overline{\text{FP}_{B=0} \square_{\text{ret}}^{-1} [\hat{n}_L S(r, t - r/c)]} = \hat{\partial}_L \left\{ \frac{\mathcal{G}(t - r/c) - \mathcal{G}(t + r/c)}{r} \right\} + \text{FP}_{B=0} \square_{\text{inst}}^{-1} [\hat{n}_L \overline{S(r, t - r/c)}], \quad (3)$$

where \hat{n}_L denotes the symmetric trace-free part of $n_L \equiv x^{i_1} \dots x^{i_\ell} / r^\ell$ ($\ell \in \mathbb{N}$). The first term is a homogeneous solution of the wave equation which is of retarded-minus-advanced type and regular at $r = 0$. It may be directly expanded in the near-zone, where it is valid by virtue of a matching argument. The second term in Eq. (3) is a particular solution of the inhomogeneous equation that is defined by means of the “instantaneous” inverse box operator, $\square_{\text{inst}}^{-1}$, representing the PN expansion of the symmetric integral operator supplemented with a regulator r^B multiplying the source and a finite part as $B \rightarrow 0$. This term diverges when $r \rightarrow 0$ and should be matched to a full-fledge solution of the field equations inside the source. However, we proved¹³ (see the appendix there) that it cannot actually contribute at half-integral PN orders, so that the effect we are looking for comes only from the term containing $\mathcal{G}(u)$. The latter function is given by a specific double integral over the source piece $S(r, t)$, and regularized by the finite part as $B \rightarrow 0$,

$$\mathcal{G}(u) = \text{FP}_{B=0} \int_{-\infty}^u ds \, R_B \left(\frac{u-s}{2}, s \right), \quad (4)$$

$$\text{where } R_B(\rho, s) = 2^{\ell-1} \rho^\ell \int_0^\rho d\lambda \frac{(r-\lambda)^\ell}{\ell!} \lambda^{B-\ell+1} S(\lambda, s). \quad (5)$$

This function is the crucial object to investigate for the purpose here.

At linear order in the mass ratio, we may disregard any hereditary term involving the product of more than two moments other than the mass monopole M , since each such multipole is proportional to ν . General results on the structure of the gravitational field in harmonic gauge tell us that hereditary contributions of type $M \times \dots \times M \times \mathcal{M}_P$, with $\mathcal{M}_P = I_P$ or J_P , read⁶

$$h_{M \times \dots \times M \times \mathcal{M}_P}^{\mu\nu} \sim \sum_{\substack{k, p, \ell, i \\ \text{integers}}} \frac{G^k M^{k-1}}{c^{3k+p}} \hat{n}_L \left(\frac{r}{c} \right)^{\ell+2i} \int_{-\infty}^{+\infty} du \, \kappa_{LP}^{\mu\nu}(t, u) \mathcal{M}_P^{(a)}(u), \quad (6)$$

where the upper sign (a) refers to time derivatives and the tensor function $\kappa_{LP}^{\mu\nu}(t, u)$ is a dimensionless kernel. Using dimensional analysis combined with “angular-momentum” selection rules, it is straightforward to show that only interactions with $k \geq 3$ can produce the half-integral PN terms of interest, starting at the leading 5.5PN order. In fact, at the next-to-next-to-leading order beyond 5.5PN, we can restrict ourselves to hereditary cubic interactions $M \times M \times \mathcal{M}_P$, which may be interpreted physically as gravitational-wave tails of tails. They must be computed at the 5.5PN, 6.5PN and 7.5PN orders for the mass quadrupole, 6.5PN and 7.5PN orders for the mass octupole and current quadrupole, and so on. The leading contributions of current-type moments are 1PN order higher than those of the mass moments.

3.2 Sketch of the calculation of u_1^T

The elementary source terms $\hat{n}_L S(r, t - r/c)$ for the tails of tails can be either instantaneous, with $S(r, t - r/c) = r^k \mathcal{M}_P^{(a)}(t - r/c)$, or hereditary. In the latter case, $S(r, t - r/c)$ is an integral of the type $r^{-k} \int_1^{+\infty} dx \, Q_m(x) \mathcal{M}_P^{(a)}(t - rx/c)$, with $Q_m(x)$ being a Legendre function of the second kind. After some transformations, the tail-of-tail piece of $\mathcal{G}(u)$ can be written as the finite part at $B = 0$ of some coefficient $C_{\ell km}(B)$, times an integral of the variable τ whose integrand involves the regulator τ^B times derivatives of $\mathcal{M}_P(t - \tau)$. Now, we find that $C_{\ell km}(B)$ may comprise (simple) poles at the order we are working, so that the factor τ^B generates a logarithm kernel $\ln \tau$. Insertion of the former piece of \mathcal{G} into the homogeneous wave entering Eq. (3) yields the form (6), with $\kappa_{LP}^{\mu\nu}(t, u)$ proportional to $\ln(t - u)$ for $t > u$, and zero elsewhere^{13,14}.

At this stage, it is important to realize that this sort of “pure” tail-of-tail contributions can generate another kind of half-integral terms, “mixed” contributions, by coupling to the non-tail part of the PN metric in the source of Einstein’s equations. This part is obtained most conveniently by solving the relaxed field equations in the near zone, where $\Lambda^{\mu\nu}$ is augmented by the matter source $16\pi Gc^{-4}|g|T^{\mu\nu}$. It is usually parametrized by means of appropriate potentials, such as the Newtonian potential $U = Gm_1/r_1 + Gm_2/r_2$ (with $r_1 = |\mathbf{x} - \mathbf{y}_1|$). The time component h^{00} of the gravitational field, for instance, is composed of “ordinary” PN terms: $-4U/c^2 + \dots$, plus tail terms containing our effect: $h_{\text{tail}(5.5\text{PN})}^{00} + \dots$. Its product with, say, h^{ij} , whose structure is similar, produces couplings that must be crucially taken into account in the calculation. Their number is minimized by moving to an adapted gauge. Quadratic and cubic PN iterations are then required to find the complete half-integral PN part of the metric at the 7.5PN order. The successive solutions are constructed by means of hierarchies of “superpotentials” derived from the potentials that enter $g_{\mu\nu}$ at the 2PN order¹⁴. In the end, we decompose the tail integrals into conservative time symmetric and dissipative time anti-symmetric pieces and simply discard the dissipative piece from our results.

Using the standard stress-energy tensor for point particles to model the binary, all multipole moments, potentials and superpotentials can be evaluated explicitly for circular orbits. The hereditary integrals are derived from standard formulas. This yields for the redshift^{13,14}

$$u_{\text{SF}}^T = -y - 2y^2 - 5y^3 + \dots - \frac{13696}{525}\pi y^{13/2} + \frac{81077}{3675}\pi y^{15/2} + \frac{82561159}{467775}\pi y^{17/2} + \dots, \quad (7)$$

where we have written only the relative 2PN terms relevant to our next-to-next-to-leading order calculation, with all the other terms indicated by ellipsis. The result (7) is in full agreement with those derived by gravitational SF methods, either semi-analytical or purely analytical^{12,16}. We emphasize that it has been achieved from the standard PN approach, which is not tuned to a particular type of source (contrary to various analytical and numerical SF calculations), as it is actually applicable to any extended PN source with compact support. The 7.5PN order reached by the present calculation is perhaps the highest order ever reached by traditional PN methods. Note also that while SF results may be relativistic but with $q \ll 1$, the present method, valid primarily in the PN regime, is in principle applicable for arbitrary mass ratios. The time is now ripe for the SF approach to proceed to second order in q . We look forward to an occasion when high precision SF methods applied at second order may be fruitfully compared with known PN results in the weak field regime.

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